Flat anisotropic models of the universe with torsion and without singularity

1975 J. Phys. A: Math. Gen. 8 L29

(http://iopscience.iop.org/0305-4470/8/3/001)

J. Phys. A: Math. Gen., Vol. 8, No. 3, 1975. Printed in Great Britain. © 1975

LETTER TO THE EDITOR

Flat anisotropic models of the universe with torsion and without singularity

B Kuchowicz

Department of Radiochemistry and Radiation Chemistry, University of Warsaw, 02-089 Warszawa, ul Zwirki i Wigury 101, Poland

Received 13 December 1974

Abstract. The intrinsic angular momentum of the cosmological substratum is able to prevent the appearance of the cosmological singularity in the Einstein–Cartan theory. Conditions for this to occur in Bianchi type I cosmologies are given, and several new cosmological models whose anisotropy diminishes with expansion are explicitly constructed.

Perhaps the most natural extension of the ideas of general relativity is to be found in the Einstein–Cartan (EC) theory which was first proposed by Cartan (1923), and has recently been developed by Trautman (1973a, b) and others; a complete bibliography to this problem may be found in the survey of Hehl (1973). The differentiable manifold in this theory has an asymmetric affine connection; the antisymmetric part of this connection is the torsion tensor which is algebraically related to the spin density tensor. The introduction of torsion is here equivalent to an introduction of extremely short-range repulsive forces which are able to prevent in principle the cosmological singularity. This was shown first by Kopczyński (1972) for the unphysical situation of a spherically symmetric distribution of spinning dust; later it was shown that non-singular cosmological models of the Bianchi I to VIII types are possible (Tafel 1973). Though the unphysical assumption of a spherical symmetry for the metric tensor was abandoned in the latter case, the results were obtained still with the help of the assumption of spin conservation which is only a sufficient condition for the validity of the generalized Bianchi identities in the EC theory (Kuchowicz 1975), and which unnecessarily restricts the classes of admissible solutions. In addition, it may prove useful to derive exact solutions for cosmological models in order to show that the admissible classes are not empty. In the following, with a metric of the form:

$$ds^2 = -X^2(t)\,dx^2 - Y^2(t)\,(dz^2+dy^2)+dt^2, \tag{1}$$

general results for Bianchi type I models are to be derived when the classical description of spin is used. This description presents the three-index tensor of spin $S^i{}_{jk}$ in terms of the standard tensor of the density of angular momentum S_{jk}: $S^i{}_{jk} = S_{jk}u^i$. Here u^i is the velocity vector of matter, and we have $u^iu_i = 1$, $S_{jk}u^k = 0$. We use co-moving coordinates, and the x axis is the spin-alignment axis, so there remains the only spin density tensor component $S_{23} = -S_{32}$, which is linearly expressed through the torsion tensor component $Q^4{}_{23}$. (The index assignments to the coordinates are obvious: $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = t$.) The energy–momentum tensor is that of a perfect fluid, and it is useful to define a main-scale factor $R(t)$ with the help of the volume expansion θ of the fluid lines:

$$\frac{\dot{R}}{R} = \frac{1}{3}\theta. \tag{2}$$

Here, as in the following, a dot denotes differentiation with respect to the time derivative. $R(t)$ corresponds to a generalization of the radius function in homogeneous and isotropic (Robertson–Walker) models, and for our metric (1) it is equal to $R(t) = (X(t)Y^2(t))^{1/3}$.

The system of EC equations

$$\begin{aligned} 8\pi G\rho &= \left(\frac{\dot{Y}}{Y}\right)^2 + 2\frac{\dot{X}\dot{Y}}{XY} + (4\pi G S_{23})^2 \\ 8\pi G p &= -2\frac{\ddot{Y}}{Y} - \left(\frac{\dot{Y}}{Y}\right)^2 + (4\pi G S_{23})^2 \\ 8\pi G p &= -\frac{\ddot{Y}}{Y} - \frac{\ddot{X}}{Y} - \frac{\dot{X}\dot{Y}}{XY} + (4\pi G S_{23})^2 \end{aligned} \tag{3}$$

may be separated out into a single equation for the shear σ, and two equations yielding energy density ρ and pressure p in terms of σ, S_{23} and R and its derivatives. By equating the two expressions for p in (3) to each other we obtain the first integral which is just the shear:

$$\sqrt{3}|\sigma| = \left|\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X}\right| = \frac{C}{R^3}, \tag{4}$$

where C is a constant, and the set (3) is reduced to:

$$8\pi G\rho = 3\left(\frac{\dot{R}}{R}\right)^2 - \frac{1}{3}\frac{C^2}{R^6} + (4\pi G S_{23})^2, \qquad 8\pi G\rho = -2\frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2 - \frac{1}{3}\frac{C^2}{R^6} + (4\pi G S_{23})^2. \tag{5}$$

It is seen easily that the light velocity c is equal to unity in our units. An integration of the system of equations (5) may be performed under the assumption of a linear equation of state:

$$p = (\gamma - 1)\rho \tag{6}$$

with $1 \leqslant \gamma < 2$, as is often done in general relativity (eg Vajk and Elgroth 1970). A first integral of equation (6) is now

$$\dot{R}^2 R^{3\gamma-2} - \tfrac{1}{3}C^2 R^{3\gamma-6} + (\gamma - 2)(4\pi G)^2 \int (S_{23}(R))^2 R^{3\gamma-1}\, \mathrm{d}R = D. \tag{7}$$

The behaviour of the scale-factor $R(t)$ depends on whether the shear does vanish or not, and on the way the spin density S_{23} changes with a change of R:

$$S_{23} = \frac{S_0}{R^\delta}, \qquad S_0 = \text{constant}. \tag{8}$$

We are able to distinguish four cases of behaviour provided $S_0 \neq 0$:

(i) Shear $\sigma = 0$, and spin conservation (corresponding to $\delta = 3$ in equation (8)). A singularity never occurs. This is the case studied by Trautman (1973b).

(ii) Shear $\sigma \neq 0$, and spin conservation. No singularity occurs provided we have the condition:

$$A \stackrel{\text{def}}{=} (4\pi G S_0)^2 - C^2 > 0. \tag{9}$$

This is the case studied by Kopczyński (1973).

(iii) Shear $\sigma = 0$, and no spin conservation, ie $\delta \neq 3$. A singularity does not occur provided we have $\delta > \frac{3}{2}\gamma$.

(iv) Shear $\sigma \neq 0$, and no spin conservation. No singularity occurs if $\delta > 3$.

Cases (iii) and (iv) give new sets of solutions. A singularity is avoided each time when the contribution from the aligned spin density to the matter density and pressure overwhelms all other contributions in the initial stages of evolution of the universe. A non-conservation of the spin density may be interpreted eg as a continuing decrease of the ordering of angular momentum of galaxies (or other particles of the cosmological substratum); this would be a quite natural process.

Exact solutions of equation (7) with spin conservation may be obtained easily for the two sequences of values of the parameter γ: $\gamma = 1 + n/(n+1)$, and $\gamma = 2 - 2/(2n+3)$. An exact solution for a dust universe ($\gamma = 1$) was presented by Kopczyński (1973); below we give the solutions for some other values of γ.

(*a*) $\gamma = \frac{4}{3}$ (corresponding to a radiation-filled universe):

$$[(DR^2 - 4A)^{1/2} + D^{1/2}R]^{4A} \exp[(DR^2 - 4A)^{1/3} D^{1/2} R] = e^{2t} \tag{10}$$

(*b*) $\gamma = \frac{3}{2}$

$$(DR^{3/2} - \tfrac{1}{3}A)^{1/2}[DR^{3/2} + \tfrac{2}{3}A] = \tfrac{9}{4}D^2t^2 \tag{11}$$

(*c*) $\gamma = \frac{5}{3}$

$$(DR - \tfrac{1}{3}A)^{1/2}\left(\frac{A^2}{9} + \frac{2A}{9}(DR - \tfrac{1}{3}A) + \tfrac{1}{5}(DR - \tfrac{1}{3}A)^2\right) = \tfrac{1}{2}D^3 t. \tag{12}$$

The solutions for $\gamma > \frac{4}{3}$ are those for an ultrarelativistic equation of state; we see that a singularity can be prevented in them. But for extremely stiff matter ($\gamma = 2$) a singularity cannot be avoided in this way, as the situation is reduced back to that of general relativity (the spin-induced terms in the equation of state cancel).

An exact solution in the case of spin non-conservation is obtained for $\delta = 3\gamma$:

$$(DR^{3\gamma} - B)^{1/2}(DR^{3\gamma} + 2B) = \tfrac{9}{2}D^2\gamma t, \qquad B = \frac{(2-\gamma)(4\pi G S_0)^2}{3\gamma} > 0. \tag{13}$$

It is characterized also by a minimum nonzero critical radius of the universe along the lines of Trautman's reasoning (Trautman 1973b).

References

Cartan E 1923 *Ann. ec. Norm. Sup.* **40** 325–412
Hehl F W 1973 *Gen. Rel. Grav.* **4** 333–50
Kopczyński W 1972 *Phys. Lett.* **39A** 219–20
—— 1973 *Phys. Lett.* **43A** 63–4
Kuchowicz B 1975 *Acta Phys. Polon.* B **6** in the press
Tafel J 1973 *Phys. Lett.* **45A** 341–2
Trautman A 1973a *Instit. Naz. Alta Mat. Symp. Mat.* **12** 139–62
—— 1973b *Nat. Phys. Sci.* **242** 7–8
Vajk J P and Elgroth P G 1970 *J. Math. Phys.* **11** 2212–22